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Research Article

A Note on Small Amplitude Limit Cycles of Liénard Equations Theory

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In this paper, we demonstrate using a counterexample for a theorem of the small amplitude limit cycles in some Liénard systems and show that that there will be no solutions unless we add an extra condition. A new condition is derived for some specific Liénard systems where a violation of the small amplitude limit cycles theorem takes place.

1. Introduction

A lot of previous works consider studies on a limit cycles' existence for Liénard systems [1–3]. It represents a very important class of nonlinear systems due to its appearance in some branches of science and engineering as well as in some ecological models, planar physical models, and even in some chemical models, where using a suitable transformation can change these systems into nonlinear Liénard systems. However, an extensive attention has been also devoted to the question of its uniqueness [4–6]; this uniqueness can be verified using different ways of methods based on Poincare–Bendixson theorem. In [4], Zhou *et al*. proposed a set of theorems for the limit cycles' uniqueness for the Liénard systems; the proposed theorems represent a guarantee to complete the proof of some previous works' propositions. In [7], Sabatini and Gabriele studied the uniqueness of limit cycles for a class of planar dynamical systems taking into account those which are equivalent to Liénard systems, and they have also proved a theorem for limit cycles of a class of plane differential systems. In the paper proposed by Li and Llibre [8], the authors proved that for any classical Liénard differential equation of degree four, there exists at most one hyperbolic limit cycle. In [9], a sufficient condition for the existence and the uniqueness of limit cycles for Liénard systems has been proposed for some applications.

In the theory of small amplitude limit cycles, Liénard systems have n solutions, However, in this paper, we use a counterexample to demonstrate that the existence of n solutions for some systems is not true unless we add an extra new condition.

We consider in our study the systems given by the following form:

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = -g(x), \end{cases} \tag{1}$$

where F and g are polynomials of order n of x and y. For several classes of such systems and in cases where the critical point is under perturbation of the coefficients in F and g, the maximum number of limit cycles that can bifurcate out can be formulated in terms of the degree of F and g [10–12].

2. Bendixon Criterion

We consider the following autonomous system:

$$\begin{cases} \dot{x} = P(x, y), \\ \dot{y} = Q(x, y). \end{cases} \quad (2)$$

Let $X = (P, Q)$ be the vector field and $\operatorname{div} X = \partial P/\partial x + \partial Q/\partial y$.

Theorem 1. *Let D be a simply connected open subset of $\mathbb{R}$. If $\operatorname{div} X = \partial P/\partial x + \partial Q/\partial y$ is of constant sign and not identically zero in D, then the system defined by 2 has no periodic orbit lying entirely in the region D.*

Proof. If γ is a periodic orbit in D, then $P(x, y)\mathrm{d}y - Q(x, y)\mathrm{d}x = 0$ on γ. Since the interior U of γ is simply connected, we can apply Greenâs theorem to obtain the following:

$$0 = \oint (P(x, y)\mathrm{d}y - Q(x, y)\mathrm{d}x) = \iint \left(\frac{\partial P}{\partial x}(x, y) + \frac{\partial Q}{\partial y}(x, y)\right)\mathrm{d}x\mathrm{d}y. \quad (3)$$

This is a contradiction since our hypothesis implies that the integral on the right cannot be zero. □

Proof. If we suppose that the system given by 2 has a periodic solution of a period T, then it has a closed orbit Γ in D. Let G be the interior of Γ, we can apply Greens theorem to obtain the following:

$$\begin{aligned} I &= \iint_G \left(\frac{\partial P}{\partial x}(x, y) + \frac{\partial Q}{\partial y}(x, y)\right)\mathrm{d}x\mathrm{d}y \\ &= \oint_\Gamma P(x, y)\mathrm{d}y - Q(x, y)\mathrm{d}x, \\ &= \int_0^T \left(P(x(t), y(t))\frac{\mathrm{d}y}{\mathrm{d}t} - Q(x(t), y(t))\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t \\ &= \int_0^T (P(x(t), y(t))Q(x(t), y(t)) \\ &\quad - Q(x(t), y(t))P(x(t), y(t)))\mathrm{d}t = 0. \end{aligned} \quad (4)$$

Since $\operatorname{div}X$ is either >0 or <0, then $\iint_G \operatorname{div}X\,\mathrm{d}x\mathrm{d}y$ will not be zero; therefore, there are no periodic solutions. □

3. A Note on Liénard Equations Theory

We consider the following system:

$$\begin{cases} \dot{x} = \left(y + c_2 y^2 + \cdots + c_L y^L\right) - \left(a_1 x + a_3 x^3 + \cdots + a_{2n+1} x^{2n+1}\right), \\ \dot{y} = -\left(x + b_2 x^2 + \cdots + b_N x^N\right), \end{cases} \quad (5)$$

where $c_2, c_3, \ldots, c_L, a_1, a_3, \ldots, a_{2n+1}, b_2, b_3, \ldots$ and b_N are real coefficients.

Theorem 2 (see [1]). *For the system of form (2), there are at most n small-amplitudes limit cycles. If $a_1, a_3, \ldots, a_{2n+1}$ are so chosen that*

$$\begin{gathered} |a_1| \ll |a_3| \ll \cdots \ll |a_{2n+1}|, \\ a_{2j-1}a_{2j+1} < 0 \ (j = 1, \ldots, n), \end{gathered} \quad (6)$$

then there are exactly n small-amplitudes limit cycles.

Proof. (counterexample).

We suppose the following system:

$$\begin{cases} \dot{x} = X = \phi(y) - F(x), \\ \dot{y} = Y = -g(x), \end{cases} \quad (7)$$

where

$$\begin{aligned} F(x) &= \sum_{j=0}^{n} \frac{(-1)^j}{(2j+1)10^{10(n-j)}} x^{2j+1}, \quad n = 2m, \\ g(x) &= x + b_2 x^2 + \cdots + b_N x^N, \\ \phi(y) &= y + c_2 y^2 + \cdots + c_L y^L. \end{aligned} \quad (8)$$

By putting $a_{2j+1} = (-1)^j/(2j+1)10^{10(n-j)}$, we obtain

$$\left|\frac{a_{2j-1}}{a_{2j+1}}\right| = \frac{1/(2j-1)10^{10(n-j+1)}}{1/(2j+1)10^{10(n-j)}} = \frac{2j+1}{2j-1}10^{-10}. \quad (9)$$

As

$$\left|\frac{a_{2j-1}}{a_{2j+1}}\right| = \frac{2j+1}{2j-1}10^{-10} \leq 3 \times 10^{-10} \ll 1, \quad \text{for } j = 1, \ldots, n, \quad (10)$$

then

$$\begin{gathered} |a_1| \ll |a_3| \ll \cdots \ll |a_{2n+1}|, \\ a_{2j-1}a_{2j+1} < 0, \text{ for } j = 1, \ldots, n. \end{gathered} \quad (11)$$

However,

$$\begin{aligned} \operatorname{div}(X, Y) &= \frac{\partial\left(\phi(y) - \sum_{j=0}^{n} a_{2j+1} x^{2j+1}\right)}{\partial x} + \frac{\partial(-g(x))}{\partial y}, \\ &= -\sum_{j=0}^{n} \frac{(-1)^j}{10^{10(n-j)}} x^{2j} \\ &= -\prod_{j=0}^{n/2-1} \left(\left(x^2 - \frac{1}{10^{10}}\cos\frac{2j+1}{n+1}\pi\right)^2 + \frac{1}{10^{20}}\sin^2\frac{2j+1}{n+1}\pi\right) < 0, \end{aligned} \quad (12)$$

because

$$\begin{aligned} f(x) &= \sum_{j=0}^{n} \frac{(-1)^j}{10^{10(n-j)}} x^{2j}, \\ &= \prod_{j=0}^{n/2-1} \left(x - 10^{-5} e^{i(2j+1)\pi/2n+2}\right)\left(x - 10^{-5} e^{-i(2j+1)\pi/2n+2}\right)\left(x + 10^{-5} e^{i(2j+1)\pi/2n+2}\right)\left(x + 10^{-5} e^{-i(2j+1)\pi/2n+2}\right) \\ &= \prod_{j=0}^{n/2-1} \left(\left(x^2 - \frac{1}{10^{10}} \cos \frac{2j+1}{n+1}\pi\right)^2 + \frac{1}{10^{20}} \sin^2 \frac{2j+1}{n+1}\pi\right). \end{aligned} \tag{13}$$

□

Theorem 3 (see [2]). *We consider the following equation:*

$$\dot{r} = r\left(v_0 + v_1 r^2 + v_2 r^4 + \cdots + v_n r^{2n}\right). \tag{14}$$

If the focus values v_j given in equation (3) satisfy the following conditions:

$$v_j v_{j+1} < 0, \text{ and } \left|v_j\right| \ll \left|v_{j+1}\right| \ll 1, \quad \text{for } j = 0, 1, 2, \ldots, n-1, \tag{15}$$

then the polynomial equation given by $\dot{r} = 0$ in equation (3) has n positive real roots for r^2.

Proof. (counterexample).

We consider the following equation:

$$\dot{r} = f(r) = r \sum_{j=0}^{n} \frac{(-1)^j}{10^{10(n-j)+10}} r^{2j}. \tag{16}$$

By putting $v_j = (-1)^j / 10^{10(n-j)+10}$, we obtain

$$v_j v_{j+1} < 0, \quad \text{and } \left|v_j\right| \ll \left|v_{j+1}\right| \ll 1, \text{ for } j = 0, 1, 2, \ldots, n-1, \tag{17}$$

because

$$\frac{v_j}{v_{j+1}} = \frac{10^{10(n-j-1)+10}}{10^{10(n-j)+10}} = 10^{-10} \ll 1, \quad \text{for } j = 0, 1, 2, \ldots, n-1, \text{ and } \left|v_n\right| = 10^{-10} \ll 1. \tag{18}$$

However,

$$\begin{aligned} \frac{f(r)}{r} &= \sum_{j=0}^{n} \frac{(-1)^j}{10^{10(n-j)+10}} r^{2j} \\ &= 10^{-10} \prod_{j=0}^{n/2-1} \left(\left(r^2 - \frac{1}{10^{10}} \cos \frac{2j+1}{n+1}\pi\right)^2 + \frac{1}{10^{20}} \sin^2 \frac{2j+1}{n+1}\pi\right) \neq 0, \quad \forall r \in \mathbb{R}. \end{aligned} \tag{19}$$

□

4. Examples

In this section, by using the counterexample, we can demonstrate that Theorems 2 and 3 are not true. However, the previous theorems will be true if we add the following condition: $a_0/a_2 \ll a_2/a_4 \ll a_4/a_6 \ll a_6/a_8 \ll \cdots \ll a_{2j-2}/a_{2j}$, $j = 1, \ldots, n$.

Example 1. We consider the following equation:

$$\dot{r} = f(r),$$

or

$$f(r) = \sum_{j=0}^{4} v_j r^{2j+1} = r\left(\frac{1}{10^{50}} - \frac{1}{10^{40}}r^2 + \frac{1}{10^{30}}r^4 - \frac{1}{10^{20}}r^6 + \frac{1}{10^{10}}r^8\right). \tag{20}$$

We have

$$v_j v_{j+1} < 0, \text{ and } \left|v_j\right| \ll \left|v_{j+1}\right| \ll 1, \quad \text{for } j = 0, 1, \ldots, 3, \tag{21}$$

because

$$\frac{v_j}{v_{j+1}} = 10^{-10} \ll 1, \quad \text{for } j = 0, 1, \ldots, 3, \text{ and } \left|v_4\right| = 10^{-10} \ll 1. \tag{22}$$

However,

$$\begin{aligned}
\frac{f(r)}{r} = 10^{-10}&\left(r - \frac{1}{10^5}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)\right)\left(r - \frac{1}{10^5}\left(\cos\frac{3\pi}{10} + i\sin\frac{3\pi}{10}\right)\right),\\
&\left(r - \frac{1}{10^5}\left(\cos\frac{7\pi}{10} + i\sin\frac{7\pi}{10}\right)\right)\left(r - \frac{1}{10^5}\left(\cos\frac{9\pi}{10} + i\sin\frac{9\pi}{10}\right)\right)\\
&\left(r - \frac{1}{10^5}\left(\cos\frac{11\pi}{10} + i\sin\frac{11\pi}{10}\right)\right)\left(r - \frac{1}{10^5}\left(\cos\frac{13\pi}{10} + i\sin\frac{13\pi}{10}\right)\right)\\
&\left(r - \frac{1}{10^5}\left(\cos\frac{17\pi}{10} + i\sin\frac{17\pi}{10}\right)\right)\left(r - \frac{1}{10^5}\left(\cos\frac{19\pi}{10} + i\sin\frac{19\pi}{10}\right)\right) \neq 0, \quad \forall r \in \mathbb{R},
\end{aligned} \tag{23}$$

where the system roots are given by

$$\begin{aligned}
r_1 &= \frac{1}{10^5}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right) = 9.5106 \times 10^{-6} + 3.0902 \times 10^{-6}i,\\
r_2 &= \frac{1}{10^5}\left(\cos\frac{3\pi}{10} + i\sin\frac{3\pi}{10}\right) = 5.8779 \times 10^{-6} + 8.0902 \times 10^{-6}i,\\
r_3 &= \frac{1}{10^5}\left(\cos\frac{7\pi}{10} + i\sin\frac{7\pi}{10}\right) = -5.8779 \times 10^{-6} + 8.0902 \times 10^{-6}i,\\
r_4 &= \frac{1}{10^5}\left(\cos\frac{9\pi}{10} + i\sin\frac{9\pi}{10}\right) = -9.5106 \times 10^{-6} + 3.0902 \times 10^{-6}i,\\
r_5 &= \frac{1}{10^5}\left(\cos\frac{11\pi}{10} + i\sin\frac{11\pi}{10}\right) = -9.5106 \times 10^{-6} - 3.0902 \times 10^{-6}i,\\
r_6 &= \frac{1}{10^5}\left(\cos\frac{13\pi}{10} + i\sin\frac{13\pi}{10}\right) = -5.8779 \times 10^{-6} - 8.0902 \times 10^{-6}i,\\
r_7 &= \frac{1}{10^5}\left(\cos\frac{17\pi}{10} + i\sin\frac{17\pi}{10}\right) = 5.8779 \times 10^{-6} - 8.0902 \times 10^{-6}i,\\
r_8 &= \frac{1}{10^5}\left(\cos\frac{19\pi}{10} + i\sin\frac{19\pi}{10}\right) = 9.5106 \times 10^{-6} - 3.0902 \times 10^{-6}i.
\end{aligned} \tag{24}$$

Example 2. Let us consider now the following system:

$$\dot{r} = f(r) = r\sum_{j=0}^{4}\frac{(-1)^j}{10^{4^{(4-j)+1}}}r^{2j} = r\left(\frac{1}{10^{1024}} - \frac{1}{10^{256}}r^2 + \frac{1}{10^{64}}r^4 - \frac{1}{10^{16}}r^6 + \frac{1}{10^4}r^8\right), \tag{25}$$

with positive roots such as

$$\begin{pmatrix} r_1 = 10^{-6} \\ r_2 = 10^{-24} \\ r_3 = 10^{-96} \\ r_4 = 10^{-384} \end{pmatrix}, \tag{26}$$

because

$$v_j v_{j+1} < 0, \quad \text{and } \left|v_j\right| \ll \left|v_{j+1}\right| \ll 1, \text{ for } j = 0, 1, \ldots, 3,$$
$$\frac{v_j}{v_{j+1}} \ll \frac{v_{j+1}}{v_{j+2}} \ll 1, \quad \text{for } j = 0, \ldots, 2. \tag{27}$$

Example 3. We suppose the following system:

$$\begin{cases} \dot{x} = y - \varepsilon\left(b_1 x + b_3 x^3 + b_5 x^5 + b_7 x^7 + b_9 x^9\right), \\ \dot{y} = -x, \end{cases}$$

or

$$\begin{aligned} b_1 &= 2\left(10^{-196}\right), \\ b_3 &= -\frac{8}{3}\left(10^{-192} + 10^{-152} + 10^{-132} + 10^{-112}\right), \\ b_5 &= \frac{16}{5}\left(10^{-148} + 10^{-128} + 10^{-110} + 10^{-88} + 10^{-68} + 10^{-48}\right), \\ b_7 &= -\frac{128}{35}\left(10^{-4} + 10^{-44} + 10^{-64} + 10^{-84}\right), \\ b_9 &= \frac{256}{63}. \end{aligned} \tag{28}$$

By putting $a_{2j+1} = \varepsilon b_{2j+1}$, $j = \overline{0,4}$, we obtain

$$\begin{cases} \dot{x} = y - \left(a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 + a_9 x^9\right), \\ \dot{y} = -x, \end{cases} \tag{29}$$

and by applying the first-order averaging method [13, 14] on (14), we obtain

$$\begin{aligned} f^0(r) = r\{&r^8 - \left(10^{-4} + 10^{-44} + 10^{-64} + 10^{-84}\right)r^6 + \left(10^{-148} + 10^{-128} + 10^{-110} + 10^{-88} + 10^{-68} + 10^{-48}\right)r^4 \\ &- \left(10^{-192} + 10^{-152} + 10^{-132} + 10^{-112}\right)r^2 + 10^{-196}\}. \end{aligned} \tag{30}$$

$f^0(r) = 0$ implied $r_1 = 10^{-2}$, $r_2 = 10^{-22}$, $r_3 = 10^{-32}$, and $r_4 = 10^{-42}$, then there are exactly 4 small-amplitudes limit cycles r_i, $i = \overline{1,4}$

Note that $a_{2j+1}/a_{2j+3} = b_{2j+1}/b_{2j+3} \ll 1$ for $j = 0, \ldots, 3$ and $b_{2j+1}/b_{2j+3} \ll b_{2j+3}/b_{2j+5}$ for $j = 0, \ldots, 2$.

5. Conclusion

In this work, by using a counterexample for a theorem of the small amplitude limit cycles in some Liénard systems, we have shown that that there will be no solutions unless an

extra condition is added. In addition, a new condition is derived for some specific Liénard systems where a violation of the small amplitude limit cycles theorem takes place. However, these theorems will be true if we add the following condition: $a_0/a_2 \ll a_2/a_4 \ll a_4/a_6 \ll a_6/a_8 \ll \cdots \ll a_{2j-2}/a_{2j}$, $j = 1, \ldots, n$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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